A specimen of a singular transformation of series*

Leonhard Euler

§1 I considered this series

$$s = 1 + \frac{ab}{1 \cdot c}x + \prod \frac{(a+1)(b+1)}{2 \cdot (c+1)}x^2 + \prod \frac{(a+2)(b+2)}{3 \cdot (c+2)}x^3 + \text{etc.}$$

where in usual manner Π denotes the coefficient of the preceding term. This series has the property that its sum seems to be impossible to exhibit in general, although it terminates and its sum is expressed in finite terms in all cases, in which either a or b is a negative number

§2 Hence if we put

$$s = z(1-x)^{c-a-b}$$

and further set

$$c - a = \alpha \quad \text{and} \quad c - b = \beta$$

the letter z will express the sum of the following series, quite similar to the preceding one,

$$z = 1 + \frac{\alpha\beta}{1 \cdot c}x + \prod \frac{(\alpha+1)(\beta+1)}{2(\alpha+1)}x^2 + \prod \frac{(\alpha+2)(\beta+2)}{3(c+2)}x^3 + \text{etc.},$$

which now terminates in all cases, in which either α or β is a positive integer, and hence if either $a - c$ or $b - c$ is a positive integer.

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§3 This transformation is to be considered to be even more important, because it seems that it cannot be found in a straight-forward way and even only by means of a differential equation of second order. Hence it will be worth one's while to have explained the whole analysis leading to this transformation.

§4 Because it is

$$s = 1 + \frac{ab}{1 \cdot c}x + \frac{ab}{1 \cdot c} \cdot \frac{(a+1)(b+1)}{2(c+1)} \cdot xx + \text{etc.},$$

and hence in each following term so the numerator as the denominator gets two new factors, let us eliminate the last two factors from each term by differentiation, which is achieved by these operations

$$\frac{\partial s}{\partial x} = \frac{ab}{1 \cdot c} + \frac{ab}{1 \cdot c}\frac{(a+1)(b+1)}{c+1}x + \text{etc.},$$

which expression multiplied by x^c and differentiated again yields

$$\partial \cdot x^c \partial s = abx^{c-1} + \frac{ab}{1 \cdot c}(a+1)(b+1)x^c + \text{etc.},$$

where we, for the sake of brevity, omitted the element ∂x, which is to remembered for the following paragraphs.

§5 Now in the same way let us add two new factors to the single numerators as follows:

1. Our series, multiplied by x^a and differentiated, will give

$$\partial \cdot x^a s = ax^{a-1} + \frac{ab}{1 \cdot c}(a+1)x^a + \text{etc.},$$

   which

2. multiplied by x^{b+1-a} and differentiated again yields

$$\partial \cdot x^{b+1-a} \partial x^a s = abx^{b-1} + \frac{ab}{1 \cdot c}(a+1)(b+1)x^b + \text{etc.},$$

which form results from the preceding one, if that one is multiplied by x^{b-c}.

§6 Hence we obtain this equation

$$\partial \cdot x^{b-a+1}\partial \cdot x^{a}s = x^{b-c}\partial \cdot x^{c}\partial s,$$

which equation in expanded form is reduced to this form

$$x^{b+1}\partial\partial s + (a+b+1)x^{b}\partial s + abx^{b-1}s = x^{b}\partial\partial s + cx^{b-1}\partial s.$$

This equation, after having divided it by x^{b-1} and having brought all terms to the right side, will obtain this form

$$0 = x(1-x)\partial\partial s + \left[c-(a+b+1)x\right]\partial s - abs$$

so that the summation of the propounded series depends on the resolution of this differential equation of second order. But this equation seems to be of such a nature that it admits no general integration.

§7 But although this differential equation does not seem to be of any use for us, it admits an extraordinary transformation completing our whole task. For, let us use this general substitution

$$s = (1-x)^{n}z,$$

whence it is

$$\log s = n\log(1-x) + \log z,$$

and by differentiation it will be

$$\frac{\partial s}{s} = \frac{\partial z}{z} - \frac{n\partial x}{1-x},$$

which equation, differentiated again, yields

$$\frac{\partial\partial s}{s} - \frac{\partial s^{2}}{ss} = \frac{\partial\partial z}{z} - \frac{\partial z^{2}}{zz} - \frac{n\partial x^{2}}{(1-x)^{2}}.$$

To this one let us add this equation

$$\frac{\partial s^{2}}{ss} = \frac{\partial z^{2}}{zz} - \frac{2n\partial x\partial z}{z(1-x)} + \frac{nn\partial x^{2}}{(1-x)^{2}}$$

and it will result

$$\frac{\partial\partial s}{s} = \frac{\partial\partial z}{z} - \frac{2n\partial x\partial z}{z(1-x)} + \frac{n(n-1)\partial x^{2}}{(1-x)^{2}}.$$

§8 Hence if the propounded equation, divided by s, is represented as follows

$$0 = x(1-x)\frac{\partial\partial s}{s} + [c-(a+b+1)x]\,\frac{\partial s}{s} - abs$$

we will, after the substitution, get to a differential equation of second order between z and x, which will be

$$\begin{aligned} & x(1-x)\frac{\partial\partial z}{z} - \frac{2nx\partial x\partial z}{z} + [c-(a+b+1)x]\,\frac{\partial z}{z} \\ & + \frac{n(n-1)x\partial x^2}{1-x} - \frac{n[c-(a+b+1)x]\partial x}{1-x} - ab = 0. \end{aligned}$$

§9 But here it is evident that the number n can be assumed in such a way that the last terms including the denominator $1-x$ can be divided by it; this happens in the case $n = -a-b+c$; having introduced this value so that it is $s = (1-x)^{c-a-b}z$, the equation between z and x will attain this form

$$x(1-x)\partial\partial z + [c+(a+b-2c-1)x]\,\partial z - (c-a)(c-b)z = 0.$$

§10 If we put $c-a = \alpha$ and $c-b = \beta$ in this equation, the equation between z and x will appear in this form:

$$x(1-x)\partial\partial z + [c-(\alpha+\beta+1)x]\,\partial z - \alpha\beta z = 0,$$

which differs from the first one only in that regard that instead of the letters a and b we here have α and β. Because the first differential equation resulted from this series

$$s = 1 + \frac{ab}{1\cdot c}x + \prod\frac{(a+1)(b+1)}{2(c+1)}xx + \prod\frac{(a+2)(b+2)}{3(c+2)}x^3 + \text{etc.},$$

vice versa from the last equation this series will result

$$z = 1 + \frac{\alpha\beta}{1\cdot c}x + \prod\frac{(\alpha+1)(\beta+1)}{2(c+1)}xx + \prod\frac{(\alpha+1)(\beta+1)}{3(c+2)}x^3 + \text{etc.},$$

while $\alpha = c-a$ and $\beta = c-b$; and these two series s and z depend on each other in such a way that it is $s = (1-x)^{c-a-b}z$ or $\frac{s}{z} = (1-x)^{c-a-b}$.

§11 But from the last differential equation one can find the same series for z by a direct method. Hence because from the first series for $x = 0$ it is $s = 1$, but we now put $z = (1-x)^{a+b-c}$, it will be $z = s = 1$ in the same case. Having noted this, let us assume a power series for z:

$$z = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.},$$

whence it is

$$\partial z = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \text{etc.}$$

and

$$\partial\partial z = 2B + 6Cx + 12Dx^2 + 20Ex^3 + 30Fx^4 + \text{etc.};$$

having substituted these expressions this will emerge

$$\begin{array}{rccccc}
x(1-x)\partial\partial z = & & 2Bx + & 6Cx^2 + & 12Dx^3 & + \text{etc.} \\
 & & - & 2Bx^2 - & 6Cx^3 & - \text{etc.} \\
c\partial z = & Ac + & 2Bcx + & 3Ccx^2 + & 4Dcx^3 & + \text{etc.} \\
-(\alpha+\beta+1)x\partial z = & & -\,(\alpha+\beta+1)Ax & -\,2(\alpha+\beta+1)Bx^2 & -\,3(\alpha+\beta+1)Cx^3 & - \text{etc.} \\
-\alpha\beta z = & -\alpha\beta - & A\alpha\beta x - & B\alpha\beta x^2 - & C\alpha\beta x^3 & - \text{etc.} \\
\hline
\end{array}$$

$$x(1-x)\partial\partial z + c\partial z - (\alpha+\beta+1)x\partial z - \alpha\beta z = 0$$

.

§12 Having put the single terms equal to zero we will obtain the following equations:

I. $Ac - \alpha\beta = 0$

II. $2B(c+1) - (\alpha+1)(\beta+1)A = 0$

III. $3C(c+2) - (\alpha+2)(\beta+2)B = 0$

IV. $4D(c+3) - (\alpha+3)(\beta+3)C = 0$

V. $5E(c+4) - (\alpha+4)(\beta+4)D = 0$

etc.

§13 Therefore, hence the same coefficients we already had, are found, namely

$$A = \frac{\alpha\beta}{1 \cdot c}$$

$$B = \frac{A(\alpha+1)(\beta+1)}{2(c+1)}$$

$$C = \frac{B(\alpha+2)(\beta+2)}{3(c+2)}$$

$$D = \frac{C(\alpha+3)(\beta+3)}{4(c+3)}$$

etc.

But because the method we used to obtain this extraordinary transformation, is quite strange and proceeds in a very non straight-forward fashion, another more direct and more natural method is still very much desired; for, the whole field of analysis would benefit immensely from it.

§14 Because in these series the number of factors grows continuously, in order to use the characters, taken from the power of a binomial, here more comfortably, let us attribute negative values to the letters a and b and in like manner also to the letters α and β by putting

$$a = -f, \quad b = -g, \quad \alpha = -\zeta \quad \text{and} \quad \beta = -\eta$$

so that it is

$$\zeta = -c - f \quad \text{and} \quad \eta = -c - g,$$

and now our two series s and z will depend on each other in such a way that it is

$$s = (1-x)^{c+f+g} z.$$

Now let us at first expand the series s in terms of these values and it will be

$$s = 1 + \frac{fg}{1 \cdot c}x + \prod \frac{(f-1)(g-1)}{2(c+1)}x^2 + \prod \frac{(f-2)(g-2)}{3(c+2)}x^3 + \text{etc.}$$

and in the same way the second series will be

$$z = 1 + \frac{\zeta\eta}{1 \cdot c}x + \prod \frac{(\zeta-1)(\eta-1)}{2(c+1)}x^2 + \prod \frac{(\zeta-2)(\eta-2)}{3(c+2)}x^3 + \text{etc..}$$

§15 Here we can already use the mentioned characters in a comfortable way. So let $\left(\frac{m}{n}\right)$ denote the coefficient of v^n in the expansion of the power of the binomial, $(1+v)^m$, so that this way we have

$$(1+v)^m = 1 + \left(\frac{m}{1}\right) v + \left(\frac{m}{2}\right) v^2 + \left(\frac{m}{3}\right) v^3 + \text{etc..}$$

Hence for the first of our series it will be $\frac{f}{1} = \left(\frac{f}{1}\right)$; further, $\frac{f(f-1)}{1\cdot 2} = \left(\frac{f}{2}\right)$; $\frac{f(f-1)(f-2)}{1\cdot 2\cdot 3} = \left(\frac{f}{3}\right)$ etc. and so this series can be represented in a shorter form as follows:

$$s = 1 + \frac{g}{c}\left(\frac{f}{1}\right) x + \frac{g}{c}\cdot\frac{g-1}{c+1}\cdot\left(\frac{f}{2}\right) x^2 + \frac{g}{c}\cdot\frac{g-1}{c+1}\cdot\frac{g-2}{c+2}\cdot\left(\frac{f}{3}\right) x^3 + \text{etc.}$$

To contract the terms containing the letter g in the same way, let us multiply the series by the character $\left(\frac{g+c-1}{c-1}\right)$ on both sides; for, it will be

$$\left(\frac{g+c-1}{c-1}\right)\frac{g}{c} = \left(\frac{g+c-1}{c}\right); \left(\frac{g+c-1}{c-1}\right)\frac{g}{c}\cdot\frac{g-1}{c+1} = \left(\frac{g+c-1}{c+1}\right),$$

which expression further multiplied by $\frac{g-2}{c+2}$ will give this character $\left(\frac{g+c-1}{c+2}\right)$. Having noted all this, we obtain this series:

$$s\left(\frac{g+c-1}{c-1}\right) = \left(\frac{g+c-1}{c-1}\right) + \left(\frac{f}{1}\right)\left(\frac{g+c-1}{c}\right) x + \left(\frac{f}{2}\right)\left(\frac{g+c-1}{c+1}\right) x^2$$
$$+ \left(\frac{f}{3}\right)\left(\frac{g+c-1}{c+2}\right) x^3 + \text{etc.}$$

§16 In like manner it will be possible to transform the other series; but there it is to be noted that this transformation can be done in two ways, depending on whether the factors of the denominators 1, 2, 3, 4 etc. are either combined with the letter ζ or with η. Therefore, at first from the first series, if we write ζ instead of f and η instead of g, we will obtain this series:

$$z\left(\frac{\eta+c-1}{c-1}\right) = \left(\frac{\eta+c-1}{c-1}\right) + \left(\frac{\zeta}{1}\right)\left(\frac{\eta+c-1}{c}\right) x + \left(\frac{\zeta}{2}\right)\left(\frac{\eta+c-1}{c+1}\right) x^2$$
$$+ \left(\frac{\zeta}{3}\right)\left(\frac{\eta+c-1}{c+2}\right) x^3 + \text{etc.}$$

But if instead of f and g in inverse order we write η and ζ, this results

$$z\left(\frac{\zeta+c-1}{c-1}\right)=\left(\frac{\zeta+c-1}{c-1}\right)+\left(\frac{\eta}{1}\right)\left(\frac{\zeta+c-1}{c}\right)x+\left(\frac{\eta}{2}\right)\left(\frac{\zeta+c-1}{c+1}\right)x^2 +\left(\frac{\eta}{3}\right)\left(\frac{\zeta+c-1}{c+2}\right)x^3+\text{etc.}$$

But for each of both cases the relation remains the same, of course

$$s=(1-x)^{c+f+g}z$$

§17 To explain more clearly how much these two series found for z differ from each other, let us write the assumed values instead of ζ and η, of course

$$\zeta=-c-f \quad \text{and} \quad \eta=-c-g$$

and the two last series for the letter z will be:

$$z\left(\frac{-g-1}{c-1}\right)=\left(\frac{-g-1}{c-1}\right)+\left(\frac{-c-f}{1}\right)\left(\frac{-g-1}{c}\right)x+\left(\frac{-c-f}{2}\right)\left(\frac{-g-1}{c+1}\right)x^2+\text{etc.}$$

$$z\left(\frac{-f-1}{c-1}\right)=\left(\frac{-f-1}{c-1}\right)+\left(\frac{-c-g}{1}\right)\left(\frac{-f-1}{c}\right)x+\left(\frac{-c-g}{2}\right)\left(\frac{-f-1}{c+1}\right)x^2+\text{etc.}$$

§18 To reduce these series to a more convenient from, let us put

$$g+c-1=h \quad \text{and} \quad c-1=e,$$

so that it is

$$c=e+1 \quad \text{and} \quad g=h-e;$$

hence our principal series will be

$$s\left(\frac{h}{e}\right)=\left(\frac{h}{e}\right)+\left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right)x+\left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right)x^2+\left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right)x^3+\text{etc.}$$

But the two following series, formed from the letter z, will be: the first

$$z\left(\frac{e-h-1}{e}\right)=\left(\frac{e-h-1}{e}\right)+\left(\frac{-e-f-1}{1}\right)\left(\frac{e-h-1}{e+1}\right)x +\left(\frac{-e-f-1}{2}\right)\left(\frac{e-h-1}{e+2}\right)x^2+\text{etc.}$$

the second

$$z\left(\frac{-f-1}{e}\right) = \left(\frac{-f-1}{e}\right) + \left(\frac{-1-h}{1}\right)\left(\frac{-f-1}{e+1}\right)x$$
$$+\left(\frac{-1-h}{2}\right)\left(\frac{-f-1}{e+2}\right)x^2 + \text{etc.};$$

but both quantities s and z depend on each other in such a way that it is

$$s = (1-x)^{f+b+1}z.$$

§19 Now let us show the great use of this transformation in a most memorable case consisting of the integral formula

$$\int \frac{\partial\varphi \cos i\varphi}{(1+aa-2a\cos\varphi)^{n+1}},$$

whose integral extended from the boundary $\varphi = 0$ to the boundary $\varphi = 180^\circ$ I conjectured to be

$$= \frac{\pi a^i}{(1-aa)^{2n+1}}V,$$

with

$$V = \left(\frac{n-1}{0}\right)\left(\frac{n+i}{i}\right) + \left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right)aa + \left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right)a^4 + \text{etc.};$$

this series, if it is compared to our principal one that it is

$$V = s\left(\frac{h}{e}\right),$$

will yield

$$h = n+i \quad \text{and} \quad e = i,$$

but then

$$f = n-i \quad \text{and} \quad x = aa;$$

therefore, the other series, formed from this, will be: the first

$$z\left(\frac{-n-1}{i}\right) = \left(\frac{-n-1}{i}\right) + \left(\frac{-n-1}{i}\right)\left(\frac{-n-1}{i+1}\right)a^2$$

$$+\left(\frac{-n-1}{2}\right)\left(\frac{-n-1}{i+2}\right)a^4+\text{etc.}$$

the second

$$z\left(\frac{i-n-1}{i}\right)=\left(\frac{i-n-1}{i}\right)+\left(\frac{-n-i-1}{1}\right)\left(\frac{i-n-1}{i+1}\right)a^2$$

$$+\left(\frac{-n-i-1}{2}\right)\left(\frac{i-n-1}{i+2}\right)a^4+\text{etc.},$$

which series results from the series V itself by writing $-n-1$ instead of n. But this relation among s and z will be

$$s=(1-aa)^{2n+1}z;$$

but then it is

$$V=s\left(\frac{n+i}{i}\right).$$

§20 Therefore, because it is

$$\int\frac{\partial\varphi\cos i\varphi}{(1+aa-2a\cos\varphi)^{n+1}}=\frac{\pi a^i}{(1-aa)^{2n+1}}V=\frac{\pi a^i}{(1-aa)^{2n+1}}\left(\frac{n+i}{i}\right)s,$$

let us write $-n-1$ instead of n in this form and let it be

$$\int\frac{\partial\phi\cos i\phi}{(1+aa-2a\cos\phi)^{-n}}\begin{bmatrix}\text{from} & \phi & = & 0^\circ\\ \text{to} & \phi & = & 180^\circ\end{bmatrix}=\frac{\pi a^i}{(1-aa)^{-2n-1}}U,$$

it will be

$$U=\left(\frac{-n-1-i}{0}\right)\left(\frac{-n-1+i}{i}\right)+\left(\frac{-n-1-i}{1}\right)\left(\frac{-n-1+i}{i+1}\right)aa+\text{etc.}$$

and therefore

$$U=z\left(\frac{i-n-1}{i}\right)=\left(\frac{i-n-1}{i}\right)(1-aa)^{-2n-1}s.$$

§21 Now let us put

$$1 + aa - 2a\cos\phi = \Delta$$

and let us consider these two values of the integrals we just obtained:

$$\text{I.} \quad \int \frac{\partial\phi \cos i\phi}{\Delta^{n+1}} = \frac{\pi a^i}{(1-aa)^{2n+1}} \left(\frac{n+i}{i}\right) s$$

$$\text{II.} \quad \int \Delta^n \partial\phi \cos i\phi = \frac{\pi a^i}{(1-aa)^{-2n-1}} \left(\frac{i-n-1}{i}\right)(1-aa)^{-2n-1} s = \pi a^i \left(\frac{i-n-1}{i}\right) s;$$

as a logical consequence we obtain this most memorable relation among these two integral formulas, extended from the boundary $\varphi = 0$ to the boundary $\varphi = 180^\circ$:

$$\int \frac{\partial\varphi \cos i\phi}{\Delta^{n+1}} : \int \Delta^n \partial \cos i\varphi = \left(\frac{n+i}{i}\right) : \left(\frac{i-n-1}{i}\right)(1-aa)^{2n+1}$$

or it will be

$$\left(\frac{n+i}{i}\right)(1-aa)^{-n} \int \Delta^n \partial\varphi \cos i\varphi = \left(\frac{-n-1+i}{i}\right)(1-aa)^{n+1} \int \Delta^{-n-1} \partial\phi \cos i\varphi.$$

§22 I had already found this last theorem some time ago by induction, and I almost despaired of its proof, and now it follows directly from the mentioned transformation of the series; hence the extraordinary use of this transformation, whose truth is to be considered to be very profound, is seen even more.

§23 But after having propounded the same theorem some time ago, the ratio of the two integral formulas given there seems to differ from the one found here quite a bit; but nevertheless they are detected to agree perfectly, if one just uses the following proportion, according to which it is in general

$$\left(\frac{n}{i}\right) : \left(\frac{-n-1}{i}\right) = \left(\frac{-n-1+i}{i}\right) : \left(\frac{n+i}{i}\right);$$

the reason for this is obvious, since it is

$$\left(\frac{-a}{i}\right) = \pm \left(\frac{a+i-1}{i}\right)$$

and therefore also

$$\left(\frac{b}{i}\right) = \pm\left(\frac{b-1+i}{i}\right),$$

where the upper signs hold, if i was an even number, the lower ones on the other hand, if i was an odd number. Hence it will be

$$\left(\frac{n+i}{i}\right) = \pm\left(\frac{-n-1}{i}\right) \quad \text{and} \quad \left(\frac{-n-1+i}{i}\right) = \pm\left(\frac{n}{i}\right).$$

§24 Hence our theorem can be stated even more conveniently. If, for the sake of brevity, we put

$$\frac{1+aa-2a\cos\phi}{1-aa} = \Theta,$$

so that it is

$$\Delta = (1-aa)\Theta$$

then this proportion will result:

$$\int \Theta^n \partial\varphi \cos i\varphi : \int \frac{\partial\varphi \cos i\varphi}{\Theta^{n+1}} = \int \frac{\Delta^n \partial \cos i\varphi}{(1-aa)^n} : \int \frac{\partial\phi \cos i\varphi (1-aa)^{n+1}}{\Delta^{n+1}}$$

$$= \left(\frac{-n-1+i}{i}\right) : \left(\frac{n+i}{i}\right) = \left(\frac{n}{i}\right) : \left(\frac{-n-1}{i}\right)$$

and so it will be

$$\left(\frac{n}{i}\right) \int \frac{\partial\varphi \cos i\varphi}{\Theta^{n+1}} = \left(\frac{-n-1}{i}\right) \int \Theta^n \partial\varphi \cos i\varphi.$$

THEOREM

§25 *If the sum of this series was known:*

$$\frac{h}{e} + \left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right)x + \left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right)x^2 + \left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right)x^3 + \text{etc.},$$

which sum we want to put $= S$*, then also the sums of the following two series can be exhibited, the first of which is this:*

$$\left(\frac{e-h-1}{e}\right) + \left(\frac{-e-f-1}{1}\right)\left(\frac{e-h-1}{e+1}\right)x + \left(\frac{-e-f-1}{2}\right)\left(\frac{e-h-1}{e+2}\right)x^2 + \text{etc.},$$

whose sum will be

$$\left(\frac{e-h-1}{e}\right)\frac{s}{\left(\frac{h}{e}\right)(1-x)^{f+h+1}},$$

where it is to be noted that it is

$$\left(\frac{e-h-1}{e}\right)=\pm\left(\frac{h}{e}\right),$$

where the upper sign holds, if i is an even number, the lower, if an odd one; hence the sum of the series is

$$\frac{\pm s}{(1-x)^{f+h+1}}$$

The other series, whose sum can be defined from this, will be

$$\left(\frac{-f-1}{e}\right)+\left(\frac{-h-1}{1}\right)\left(\frac{-f-1}{e+1}\right)x+\left(\frac{-h-1}{2}\right)\left(\frac{-f-1}{e+2}\right)x^2+\text{etc.},$$

whose sum will be

$$\left(\frac{-f-1}{e}\right)\frac{s}{\left(\frac{h}{e}\right)(1-x)^{f+h+1}},$$

which can also be expressed in this way:

$$\pm\left(\frac{f+e}{e}\right)\frac{s}{\left(\frac{h}{e}\right)(1-x)^{f+h+1}}.$$

§26 If the sums of these three series are stated as follows

$$\mathfrak{A}=\left(\frac{h}{e}\right)+\left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right)x+\left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right)x^2+\left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right)x^3+\text{etc.};$$

$$\mathfrak{B}=\left(\frac{e-h-1}{e}\right)+\left(\frac{-e-f-1}{1}\right)\left(\frac{e-h-1}{e+1}\right)x+\left(\frac{-e-f-1}{2}\right)\left(\frac{e-h-1}{e+2}\right)x^2+\text{etc.};$$

$$\mathfrak{C}=\left(\frac{-f-1}{e}\right)+\left(\frac{-h-1}{1}\right)\left(\frac{-f-1}{e+1}\right)x+\left(\frac{-h-1}{2}\right)\left(\frac{-f-1}{e+2}\right)x^2+\text{etc.},$$

they are related to each other as follows

$$\left(\frac{e-h-1}{e}\right)\mathfrak{A}=\left(\frac{h}{e}\right)(1-x)^{f+h+1}\mathfrak{B}$$

$$\left(\frac{-f-1}{e}\right)\mathfrak{A}=\left(\frac{h}{e}\right)(1-x)^{f+h+1}\mathfrak{C}$$

$$\left(\frac{-f-1}{e}\right)\mathfrak{B}=\left(\frac{e-h-1}{e}\right)\mathfrak{C}.$$